Lehmer's Conjecture on the Non-vanishing of Ramanujan's Tau Function

Will Y. Lee

Abstract

In this paper we prove Lehmer's conjecture on Ramanujan's tau function, namely $\tau(n) \neq 0$ for each $n \geq 1$ by investigating the additive group structure attached to $\tau(n)$ with the aid of unique factorization theorem.

[1]

Note from the uploader (D. Zeilberger): Prof. W. Lee requested that I upload this new version, of Oct. 26, 2014. I have not read it, and do not have the expertise to judge whether it is correct.

Let E_k $(k = 2, 4, \dots)$ be the normalized Eisenstein series ($[4 : 108 - 122]$) given by

$$E_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \tag{1}$$

where $q := e^{i2\pi z}$ $(\Im(z) > 0)$, B_k the Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$

and $\sigma_{k-1}(n)$ the divisor function:

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}.$$

For an elliptic curve given by

$$y^2 = 4x^3 - g_2(z)x - g_3(z) \tag{2}$$

[1] 2000 Mathematics Subject Classification. Primary $11L40$; Secondary $11YXX$

where $g_2(z) = 120\zeta(4)E_4(z), g_3(z) = 280\zeta(6)E_6(z)$ and $E_k(z)$ given by equation (1) and $\zeta(k)$ is Riemann zeta function:

$$\zeta(k) := \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

A simple calculation ([1 : 14], [4 : 112]) shows the discriminant $\Delta(z) := 4^4(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2$, where x_1, x_2 and x_3 are the roots the right side of equation (2), is given by

$$\Delta(z) = g_2(z)^3 - 27g_3(z)^2 = \frac{(2\pi)^{12}}{1728}(E_4(z)^3 - E_6(z)^2). \tag{3}$$

On the other hand Jacobi's theorem ([4 : 122]) asserts that

$$(2\pi)^{-12}\Delta(z) = q\prod_{n=1}^{\infty}(1 - q^n)^{24}. \tag{4}$$

From equation (4), Ramanujan has defined his tau function $\tau(n)$ ([1], [2], [3], [4 : 122], [5] – [7]) by

$$q\prod_{n=1}^{\infty}(1 - q^n)^{24} := \sum_{n=1}^{\infty}\tau(n)q^n. \tag{5}$$

Notice that each $\tau(n)$ $(n \geq 1)$ has an integer value. In a series of papers ([5] – [7]), D.H. Lehmer investigated the properties of $\tau(n)$ for $n \leq 300$, proved that $\tau(n) \neq 0$ for $n < 3316799$, later for $n < 214928639999$ ([1 : 22]). He also showed that if $\tau(n) = 0$ then n must be a prime. He then conjectured, what is nowadays known as Lehmer's conjecture ([6]) that

$$\tau(n) \neq 0 \text{ for each } n \geq 1. \tag{6}$$

A simple calculation ([3 : 21 – 22], [4 : 122 – 123]) shows

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sum_{j=1}^{n-1}\sigma_5(j)\sigma_5(n-j). \tag{7}$$

Since Lehmer's conjecture is equivalent to $3\tau(n) \neq 0$ for each $n \geq 1$, we write

$$A(n) := \frac{65}{252}\sigma_{11}(n) + \frac{691}{252}\sigma_5(n);\ B(n) := 691\sum_{j=1}^{n-1}\sigma_5(j)\sigma_5(n-j). \tag{8}$$

Then $3\tau(n) = A(n) - B(n)$. Observe that $A(n)$ takes on integer value for each $n \geq 1$ since both $\tau(n)$ and $B(n)$ do. Now Lehmer's conjecture is, in view of equations (7), (8) and the unique factorization theorem, equivalent to:

$$A(n) \neq B(n) \ \text{ for each } \ n \geq 1. \tag{9}$$

Recent calculation by Bosman confirms Lehmer's conjecture for $n \leq 22798241520242687999$. In this paper we prove equation (9) by showing that $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{k=0}^{q-1}$ forms an additive group of order q modulo q for $q \mid A(p),\ q > p,\ p \equiv -1 \bmod 691,\ [a_{i,k}]_{0\leq i,k\leq q-1}$ $q \times q$-matrix, with the aid of the unique factorization theorem, the pigeonhole principle and the remainder theorem. We prove equation (9) first for prime p then for $p^\alpha, \alpha \geq 2$ and finally for any composite number n. Since $11 \nmid 690$ and since $(p+1) \mid (p^{11}+1)$, the following Lemma 1 evidently holds.

Lemma 1 *Let $A(p)$ be given by equation* (8)*. Then the following two conditions* (10) *and* (11) *are equivalent:*

$$A(p) \equiv 0 \mod 691. \tag{10}$$

$$p \equiv -1 \mod 691. \tag{11}$$

If $691 \nmid A(p)$ or equivalently p does not satisfy equation (11) then we trivially have $A(p) \neq B(p)$ by equation (8). It suffices therefore to prove Lehmer's conjecture for prime p satisfying equation (10) or (11). In what follows, prime p satisfies either equation (10)

or (11). We first prove:

Lemma 2 *Let p satisfy equation* (10) *or* (11). *Then $A(p)$ has at least one prime factor q greater than p.*

Proof. Write $x = [x] + \{x\}$ where $[x]$ represents the integral part of x and $\{x\}$ the non-integral part of x. Express $A(p)$ from equation (8) as

$$\begin{array}{l} A(p) \\ = \frac{65}{252}(1+p^{11}) + \frac{691}{252}(1+p^5) \\ = 3 + \sum_{i=5}^{10} a_i p^i, a_{10} := [\frac{65p}{252}], a_i := [p\{65p^{10-i}\}]\ (6 \le i \le 9), a_5 := p\{\frac{65p^5}{252}\} + \frac{691}{252}. \end{array} \tag{12}$$

Notice that since $A(p)$ and a_i $(6 \le i \le 10)$ are positive integers from equations (8) and (12) respectively, $a_5 = p\{\frac{65p^5}{252}\} + \frac{691}{252}$ is also a positive integer. Now $A(p)$ either has at least one prime factor $q > p$ or $A(p)$ has no prime factor $q > p$, since $p \nmid A(p)$. If $A(p)$ has a prime factor $q > p$, we are done. Suppose $A(p)$ has no prime factor greater than p. $A(p)$ then is written via the unique factorization theorem as

$$A(p) = q_0^{e_0} q_1^{e_1} \dots q_m^{e_m},\ q_i < p,\ e_i \ge 1\ (q_0 := 2,\ 0 \le i \le m). \tag{13}$$

Notice that A(p) has an even factor 2^{e_0} $(e_0 \ge 1)$ by substituting equation (11) into equation (8). Since $p^{10} < A(p) < p^{11}$, the p-adic representation of $A(p)$ from equation (13) is uniquely given by

$$A(p) = \sum_{i=0}^{10} a_i p^i,\ a_i := [p\{\frac{A(p)}{p^{i+1}}\}]\ (0 \le i \le 10). \tag{14}$$

We show a_i given by equation (14) satisfies

$$a_i \ge 1\ (1 \le i \le 4). \tag{15}$$

Let g be a primitive root modulo p^i $(1 \le i \le 5)$. Then each q_k $(0 \le k \le m)$ is uniquely expressed as follows:

$$
\begin{aligned}
g^{u_{k,i}} &\equiv q_k \mod p^i \ (q_0 := 2,\ 0 \le k \le m,\ 1 \le i \le 5). && (16)\\
u_{k,i+1} &= \begin{cases} 1, & q_k = g \text{ for some } k\ (0 \le k \le m) \\ u_{k,i} + \delta_{k,i} p^{i-1}(p-1), & q_k \ne g\ (0 \le k \le m,\ 1 \le i \le 4). \end{cases} && (17)\\
\textstyle\sum_{k=0}^{m} e_k u_{k,i} &\equiv T_i \mod p^{i-1}(p-1)\ (1 \le i \le 5). && (18)
\end{aligned}
$$

For $q_k \ne g$ $(0 \le k \le m)$ in equation (17), $\delta_{k,i} \ge 1$ for almost all k $(0 \le k \le m)$ and i $(1 \le i \le 5)$ from equation (16) and the property of primitive root $g \bmod p^i$ $(1 \le i \le 5)$. The case $\delta_{k,i} = 0$ $(0 \le k \le m,\ 1 \le i \le 5)$ for $q_k \ne g$ $(0 \le k \le m)$ is exceptionally rare. Equations (14), (16) and (18) lead us to:

$$
g^{T_i} \equiv A(p) \mod p^i \ (1 \le i \le 5). \tag{19}
$$

Substitution of equations (16) and (17) into equation (18) shows us:

$$
\begin{array}{rlllll}
 & T_i & = T_1 & + & \sum_{k=1}^{i-1} d_k p^{(k-1)}(p-1) & (2 \le i \le 5) \\
\Leftrightarrow & T_{i+1} & = T_i & + & d_i p^{i-1}(p-1) & (1 \le i \le 4)
\end{array} \tag{20}
$$

In equation (20), we show $d_i \ge 1$ $(1 \le i \le 4)$. Suppose $d_i = 0$ $(1 \le i \le 4)$. We then have for each i $(1 \le i \le 4)$:

$$
\begin{array}{rlcll}
 & T_{i+1} & \equiv & T_i & \mod p^i(p-1) \\
\Leftrightarrow & T_{i+1} & = & T_i & \text{by (20)} \\
\Leftrightarrow & \sum_{k=0}^{m} e_k(u_{k,i+1} - u_{k,i}) & = & 0 & \text{by (18)} \\
\Leftrightarrow & u_{k,i+1} - u_{k,i} & = & 0 & \\
\Leftrightarrow & u_{k,i+1} & = & u_{k,i} & (0 \le k \le m).
\end{array} \tag{21}
$$

The last line of equation (21) contradicts equation (17). Thus the assumption $d_i = 0$ $(1 \le i \le 4)$ is false. This establishes $d_i \ge 1$ $(1 \le i \le 4)$. Since we have established $d_i \ge 1$ $(1 \le i \le 4)$, equation (20) implies

$$
T_1 < T_2 < T_3 < T_4 < T_5. \tag{22}
$$

Equations (14) and (19) imply

$$g^{T_i} \equiv \sum_{k=0}^{i-1} a_k p^k \mod p^i \ (1 \leq i \leq 5). \tag{23}$$

We prove $a_i \geq 1$ by induction on i $(1 \leq i \leq 4)$. Now

$$\begin{array}{rcll} g^{T_2} & = & g^{T_1} g^{T_2 - T_1} \text{ by (22)} & \\ & = & g^{T_1} g^{d_1 (p-1)} \text{ by (20)} & \\ & \equiv & g^{T_1}(1 + c_1 p)^{d_1} \bmod p^2 \text{ by Fermat}, & c_1 = [p\{\frac{g^{p-1}}{p^2}\}], d_1 = \frac{T_2 - T_1}{p-1} \\ & \equiv & g^{T_1}(1 + c_1 d_1 p) \bmod p^2 & \\ & \equiv & (a_0 + a_1' p)(1 + b_1 p) \bmod p^2, & b_1 \equiv c_1 d_1 \bmod p \\ & \equiv & a_0 + a_1 p \bmod p^2, & a_1 \equiv (a_0 b_1 + a_1') \bmod p. \end{array} \tag{24}$$

If $a_1' = 0$, then $a_1 \geq 1$ readily follows from equation (24) since $a_1 \equiv a_0 b_1 \not\equiv 0 \bmod p$. Notice that $a_0, b_1 \not\equiv 0 \bmod p$. Without loss of generality therefore, let $a_1' \geq 1$. Suppose $a_1 \equiv 0 \bmod p$ from equation (24), or equivalently assume $b_1 \equiv a_0^{-1}(-a_1') \bmod p$. Since $T_1 \neq \frac{(p-1)}{2}, p-1$ and since $a_0^{-1} = [p\{\frac{g^{p-1-T_1}}{p}\}]$, $-a_1' = [p\{\frac{g^{(p-1)/2+T_1}}{p^2}\}]$ and $c_1 = [p\{\frac{g^{p-1}}{p^2}\}]$ from equations (23) and (24) respectively, we have $c_1 \neq a_0^{-1}, -a_1'$. It follows that $c_1 d_1 \not\equiv a_0^{-1}(-a_1') \bmod p$, or equivalently $b_1 \neq b_1$ which is absurd. It follows the assumption $a_1 = 0$ is false. This establishes $a_1 \geq 1$. Suppose $a_{i-1} \geq 1$ $(2 \leq i \leq 4)$. Now

$$\begin{array}{ll} g^{T_{i+1}} & \\ = g^{T_i} g^{T_{i+1} - T_i} \text{ by (20)} & \\ = g^{T_i} g^{d_i p^{i-1}(p-1)} \text{ by (20)} & \\ \equiv g^{T_i}(1 + c_i p^i)^{d_i} \bmod p^{i+1} \text{ by Fermat}, & c_i = [p\{\frac{g^{p^{i-1}(p-1)}}{p^{i+1}}\}], d_i = \frac{T_{i+1} - T_i}{p^{i-1}(p-1)} \\ \equiv g^{T_i}(1 + c_i d_i p^i) \bmod p^{i+1} & \\ \equiv (a_0 + \cdots + a_{i-1} p^{i-1} + a_i' p^i)(1 + b_i p^i) \bmod p^{i+1} \text{ by I. H.}, & b_i \equiv c_i d_i \bmod p \\ \equiv a_0 + \cdots + a_{i-1} p^{i-1} + a_i p^i \bmod p^{i+1}, & a_i \equiv (a_0 b_i + a_i') \bmod p. \end{array} \tag{25}$$

As in the case of $i = 1$, the assumption $a_i' = 0$ implies $a_i \equiv a_0 b_i \not\equiv 0 \bmod p$ since $a_0, b_i \not\equiv 0 \bmod p$. Without loss of generality therefore, let $a_i' \neq 0$. Suppose $a_i = 0$ or equivalently $b_i \equiv a_0^{-1}(-a_i') \bmod p$. Since $-a_i' = [p\{\frac{g^{((p-1)/2+T_i)}}{p^{i+1}}\}]$ and $c_i = [p\{\frac{g^{p^{i-1}(p-1)}}{p^{i+1}}\}]$ from equations (23) and (25) respectively, we immediately have $c_i \neq a_0^{-1}, (-a_i')$ and hence $c_i d_i \not\equiv a_0^{-1}(-a_i') \bmod p$, or equivalently $b_i \neq b_i$ $(2 \leq i \leq 4)$ which is manifestly false. Thus the assumption $a_i = 0$ $(2 \leq i \leq 4)$ is false. Consequently $a_i \geq 1$ $(1 \leq i \leq 4)$.

This establishes equation (15). Since we have established equation (15), equation (14) contradicts equation (12), a contradiction to the uniqueness of the p-adic representation of $A(p)$. It follows the assumption $A(p)$ has no prime factor q greater than p is false. This completes the proof of Lemma 2.

In view of equation (15), since $a_i \geq 1$ $(1 \leq i \leq 4)$, subtraction of equation (14) from equation (12) with rearrangement of terms leads us to:

$$p^4 < \sum_{i=1}^{4} a_i p^i = 3 - a_0 < p. \tag{26}$$

Inequality (26), namely $p^4 < p$ is patently false. Consequently the assumption $A(p)$ has no prime factor greater than p is false. This gives a second proof for Lemma 2 without invoking the uniqueness of the p-adic representation of $A(p)$.

It is easy to check our proof for Lemma 2 works for all primes $p > 252$, say. A simple computation reveals Lemma 2 also holds for primes $p \leq 252$. It follows that Lemma 2 holds for all primes p. Thus the assumption that the prime p generated by equation (11) in Lemma 2 is redundant.

Let q be an odd prime prime factor of $A(p)$ greater than p. Lemma 2 guarantees existence of such a prime $q > p$. Construct matrix $[a_{i,k}]_{0 \leq i,k \leq q-1}$ as follows:

$$a_{i,k} := \sum_{\substack{j=1 \\ i691\sigma_5(j)\sigma_5(p-j) \equiv k \bmod q}}^{p-1} 1. \tag{27}$$

Since $\sigma_5(j)\sigma_5(p-j) = \sigma_5(p-j)\sigma_5(p-(p-j))$, we have from equation (25)

$$a_{i,k} = 2 \sum_{\substack{j=1 \\ i691\sigma_5(j)\sigma_5(p-j) \equiv k \bmod q}}^{(p-1)/2} 1. \tag{28}$$

Then the matrix $[a_{i,k}]_{0\leq i,\, k\leq q-1}$ has the following properties:

$$\begin{aligned}
a_{i,k} &\equiv 0 \mod 2 && (0\leq i,k\leq q-1). && (29)\\
a_{0,0} &= p-1,\ a_{0,k}=0 && (1\leq k\leq q-1). && (30)\\
a_{i,0} &= a_{j,0} && (1\leq i\neq j\leq q-1). && (31)\\
a_{i,k} &= a_{q-i,q-k} && (1\leq i,k\leq q-1). && (32)\\
i691\sum_{j=1}^{p-1}\sigma_5(j)\sigma_5(p-j) &\equiv \sum_{k=1}^{q-1} ka_{i,k} \mod q && (1\leq i\leq q-1). && (33)\\
\sum_{k=1}^{q-1} ka_{i,k} &\equiv i\sum_{k=1}^{q-1} ka_{1,k} \mod q && (1\leq i\leq q-1). && (34)
\end{aligned}$$

Notice that given $a_{1,k}\ (1\leq k\leq q-1),\ a_{i,k}\ (2\leq i\leq q-1, 1\leq k\leq q-1)$ are reshuffles of $a_{1,k}(1\leq k\leq q-1)$ and vice versa determined by

$$a_{i,k}=a_{1,i^{-1}k \bmod q} \Longleftrightarrow a_{1,k}=a_{i,ik \bmod q}\ \ (1\leq i,k\leq q-1). \tag{35}$$

For each $i=1,2,\ldots,q-1$, write $f_{j_i}:\equiv i691\sigma_5(j)\sigma_5(p-j) \bmod q$. Then $f_{j_i}=f_{p-j_i}$ $(1\leq i\leq q-1)$ from equation (27) or (28). Define for each $i=1,2,\ldots,q-1$:

$$\begin{aligned}
S_{i,l} &:= \{k: a_{i,k}=2l,\ 0\leq k\leq q-1,\ 0\leq l\leq q_0\}\\
\Longleftrightarrow &= \{(j_{i_1},j_{i_2},\ldots,j_{i_l}):\ f_{j_{i_1}}=f_{j_{i_2}}=\cdots=f_{j_{i_l}}=k,\ 1\leq j_{i_1}<j_{i_2}<\cdots<j_{i_l}<\tfrac{(p-1)}{2}\}\\
S_{i,l} &= \varnothing \text{ for } l>q_0.
\end{aligned} \tag{36}$$

Since $q>p$ and since $a_{i,k}\ (1\leq i,k\leq q-1)$ cannot be too large even number from equations (27) and (29), a positive integer $q_0<q-1$ exists, depending on p and q, satisfying the last line of equation (36). It is clear from equation (36) with the aid of equation (35) that for each $l=1,2,\ldots,q_0$:

$$S_{i,l}=S_{j,l}\ (1\leq i<j\leq q-1). \tag{37}$$

For each $q \mid A(p)$ with $q > p$, we then have from equations $(35) - (37)$ that

$$\begin{aligned} \textstyle\sum_{l=0}^{q_0} \mid S_{i,l} \mid &= q \ (1 \leq i \leq q-1). \qquad (38) \\ \textstyle\sum_{k=1}^{q-1} a_{i,k} &= \displaystyle\sum_{l=1}^{q_0} 2l \mid S_{i,l} \mid = p-1 \ (1 \leq i \leq q-1). \qquad (39) \end{aligned}$$

Equation (39) reads when $q \mid A(p)$ with $q < p$ that:

$$\sum_{k=0}^{q-1} a_{i,k} = p-1 \ (1 \leq i \leq q-1). \qquad (40)$$

Equations $(29) - (35)$ readily follow from equations (27) and (28). Equation (33) is a restatement of the remainder theorem in view of equations $(27), (36)$ and (39). Equations $(38), (39)$ and (40) follow from the pigeonhole principle. Since we exclusively use $S_{1,l}$ $(1 \leq l \leq q_0)$ in what follows, we show the following inequality:

$$\mid S_{1,l-1} \mid > \ l \mid S_{1,l} \mid \ (2 \leq l \leq q_0). \qquad (41)$$

To prove inequality (41) we use the second line of equation (36) for the definition of $S_{1,l}$. Consider the map $\beta : S_{1,l} \mapsto S_{1,l-1} \times S_{1,l-1} \times \cdots \times S_{1,l-1}$ given by

$$\begin{aligned} \beta(j_1, j_2, \ldots, j_l) := \ & ((\beta_1(j_1), \beta_2(j_1), \ldots, \beta_{l-1}(j_1)), (\beta_1(j_2), \beta_2(j_2), \ldots, \beta_{l-1}(j_2)), \ldots, \\ & (\beta_1(j_l), \beta_2(j_l), \ldots, \beta_{l-1}(j_l))) \end{aligned} \qquad (42)$$

such that for each $i = 1, 2, \ldots, l$:

$$\begin{aligned} \beta_1(j_i) &:= \min_{j_{i_k}} \{ \mid j_i - j_{i_k} \mid : a_{1,j_{i_k}} = 2(l-1) \} \\ f_{\beta_1(j_i)} &= f_{\beta_{2(j_i)}} = \cdots = f_{\beta_{l-1}(j_i)}, \ 1 \leq \beta_1(j_i) < \beta_2(j_i) < \cdots < \beta_{l-1}(j_i) < \tfrac{(p-1)}{2}. \end{aligned} \qquad (43)$$

Existence of $\beta_1(j_i)$ $(1 \leq i \leq l)$ follows from the definition of equation (36). In the second line of equation (43), $\beta_k(j_i)$ $(2 \leq k \leq l-1, 1 \leq i \leq l)$ are uniquely determined once $\beta_1(j_i)$ $(1 \leq i \leq l)$ is determined by the first line of equation (43). In the second line of

inequality (43), we rename $\{\beta_k(j_i)\}_{k=1}^{l-1}$ $(1 \le i \le l)$ in ascending order. Observe that each $(\beta_1(j_i), \beta_2(j_2) \ldots, \beta_{l-1}(j_l)) \in S_{1,l-1}$ $(1 \le i \le l)$ is distinct from equations (36) and (43). To show that the map $\beta : S_{1,l} \mapsto S_{1,l-1} \times S_{1,l-1} \times \cdots \times S_{1,l-1}$ given by equations (42) and (43) maps $S_{1,l}$ into a proper subset of $S_{1,l-1}$, define

$$\begin{aligned} \beta_1(j_1') &:= \min_{j_{1_k}} \{ \mid j_1 - j_{1_k} \mid : a_{1,j_{1_k}} = 2(l-1),\ \beta_1(j_1) \neq \beta_1(j_1') \} \\ f_{\beta_1(j_1')} &= f_{\beta_2(j_1')} = \cdots = f_{\beta_{l-1}(j_1')},\ 1 \le \beta_1(j_1') < \beta_2(j_1') < \cdots < \beta_{l-1}(j_1') < \tfrac{(p-1)}{2}. \end{aligned} \tag{44}$$

Existence of $\beta_1(j_1')$ follows from the definition of equation (36). Observe that $(\beta_1(j_1'), \beta_2(j_1'), \ldots, \beta_{l-1}(j_1')) \in S_{1,l-1}$ and distinct from $(\beta_1(j_i), \beta_2(j_i), \ldots, \beta_{l-1}(j_i))$ $(1 \le i \le l)$ by equations (43) and (44). In the second line of inequality (44), we rename $\{\beta_k(j_1')\}_{k=1}^{l-1}$ $(1 \le i \le l)$ in ascending order. Equations (42), (43) and (44) imply that the map $\beta : S_{1,l} \mapsto S_{1,l-1} \times S_{1,l-1} \times \cdots \times S_{1,l-1}$ given by equations (42) and (43) maps $S_{1,l}$ into a proper subset of $S_{1,l-1}$ in a fashion of 1 to l. This establishes inequality (41). Of course we can improve inequality (41) such that $\mid S_{1,l-1} \mid > 2l \mid S_{1,l} \mid$ by slightly changing the map β in equations (42) and (43) however, inequality (41) is good enough for our purpose. Inequality (41) implies a majority of nonzero $a_{i,k} = 2$ $(1 \le i, k \le q-1)$. See Table 1 for examples of primes p with $q \mid A(p),\ q > p$, satisfying inequality (41), where $q_0 \le 4$. Lehmer's conjecture therefore is equivalent via equation (33) for $i = 1$ to:

$$\sum_{k=0}^{q-1} k a_{1,k} \not\equiv 0 \mod q. \tag{45}$$

Since both $A(p)$ and $B(p)$ are even and divisible by 691, we have $(A(p), B(p)) \ge 1382$. Suppose q divides both $A(p)$ and $B(p)$. Then by equation (33), we have:

$$\sum_{k=0}^{q-1} k a_{i,k} \equiv 0 \mod q\ (0 \le i \le q-1). \tag{46}$$

Clearly equation (46) is equivalent by equation (34) to:

$$\sum_{k=0}^{q-1} k a_{1,k} \equiv 0 \mod q. \tag{47}$$

Since $\sum_{k=0}^{q-1} k a_{0,k} = 0 \equiv 0 \bmod q$ by equation (30), it follows that $\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1} = \{0\}$, the trivial additive group modulo q. Conversely, equation (46) or (47) implies both $q \mid A(p)$ and $q \mid B(p)$ by equation (33). On the other hand, since nonzero $a_{i,k}$ $(0 \leq i \leq q-1)$ is even and ≥ 2 from equation (29), with the aid of the unique factorization theorem and inequality (41) that a majority of nonzero $a_{i,k} = 2$, equation (46) or (47) is equivalent to:

$$\min_{1 \leq i < j \leq q-1} (\sum_{k=0}^{q-1} k a_{i,k}, \sum_{k=0}^{q-1} k a_{j,k}) = 2q. \tag{48}$$

Consequently equation (46), (47) or (48) completely characterizes common odd prime factors of both $A(p)$ and $B(p)$. We thus have:

Lemma 3 *Let q be a prime greater than p. The following conditions are equivalent:*

(i) *q divides both $A(p)$ and $B(p)$.*

(ii) $\sum_{k=0}^{q-1} k a_{i,k} \equiv 0 \bmod q$ $(0 \leq i \leq q-1)$.

(iii) $\sum_{k=0}^{q-1} k a_{1,k} \equiv 0 \bmod q$.

(iv) $\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1} = \{0\}$, *the trivial additive group modulo q.*

(v) $\min_{1 \leq i < j \leq q-1} (\sum_{k=0}^{q-1} k a_{i,k}, \sum_{k=0}^{q-1} k a_{j,k}) = 2q$.

Lemma 4 *(Main Lemma) Let p satisfy equation* (10) *or* (11) *and let $q \mid A(p)$ with $q > p$. Then $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ forms an additive group of order q modulo q.*

Proof. Let $a_{i,k}$ $(0 \leq i, k \leq q-1)$ be defined by equation (27). We have for each $i = 1, 2, \ldots, q-1$:

$$\begin{array}{lllll} & \sum_{k=0}^{q-1} ka_{i,k} & + & \sum_{k=0}^{q-1} ka_{q-i,k} & \\ = & \sum_{k=1}^{q-1} ka_{i,k} & + & \sum_{k=1}^{q-1} ka_{i,q-k} & \text{by (32)} \\ = & \sum_{k=1}^{q-1} ka_{i,k} & + & \sum_{k=1}^{q-1} (q-k)a_{i,k} & \\ = & q \sum_{k=1}^{q-1} a_{i,k} & & & \\ = & q \sum_{l=1}^{q_0} 2l \mid S_{1,l} \mid & \text{by} & (39) & \\ = & q(p-1) & \text{by} & (39). & \end{array} \tag{49}$$

Notice that equation (49) holds regardless of $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ being trivial or not. We claim that $\{\sum_{k=0}^{q-1} ka_{i,k}\}_{i=0}^{q-1}$ are all distinct. To show the claim, observe that $\{S_{1,l}\}_{l=0}^{q_0}$ are disjoint from equation (36). Since $a_{i,k} = a_{1,i^{-1}k \bmod q}$ from equation (35), we have with the aid of equation (36) for each $l = 1, 2, \ldots, q_0$:

$$\begin{array}{lll} \sum\limits_{k \in S_{1,l}} ka_{i,k} & = & \sum\limits_{k \in S_{1,l}} ka_{1,i^{-1}k \bmod q} \quad = \\ \sum\limits_{k \in S_{1,l}} ik \pmod q \, a_{1,k} & = & 2l \sum\limits_{k \in S_{1,l}} ik \pmod q. \end{array} \tag{50}$$

It is evident for each $1 \leq i \neq j \leq q-1$ and each l $(1 \leq l \leq q_0)$ that:

$$\sum_{k \in S_{1,l}} ik \pmod q \neq \sum_{k \in S_{1,l}} jk \pmod q. \tag{51}$$

For each $1 \leq i \neq j \leq q-1$, conjunction of equations (50) and (51) leads us to

$$\begin{array}{lll} & \sum_{k=0}^{q-1} ka_{i,k} & \\ = & \sum_{l=1}^{q_0} 2l \sum\limits_{k \in S_{1,l}} ik \pmod q & \text{by (50)} \\ \neq & \sum_{l=1}^{q_0} 2l \sum\limits_{k \in S_{1,l}} jk \pmod q & \text{by (41) \& (51)} \\ = & \sum_{k=0}^{q-1} ka_{j,k} & \text{by (50).} \end{array} \tag{52}$$

Equation (52) establishes the claim. Since $\sum_{k=0}^{q-1} ka_{1,k} \bmod q$ is a generator for the additive group $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ from equation (34) if it is nontrivial, it suffices therefore to

show that

$$\sum_{k=0}^{q-1} ka_{1,k} \not\equiv 0 \bmod q. \tag{53}$$

Write

$$C_i := \sum_{k=0}^{q-1} ka_{i,k} \ (1 \leq i \leq q-1). \tag{54}$$

Notice that $\{C_i\}_{i=1}^{q-1}$ are distinct ≥ 1 from equation (52). Rename C_i $(1 \leq i \leq q-1)$ again as C_i $(1 \leq i \leq q-1)$ in ascending order as follows:

$$1 \leq C_1 < C_2 < \cdots < C_{q-1}. \tag{55}$$

We claim that there is at least one adjacent pair $\{C_j, C_{j+1}\}$ $(1 \leq j \leq q-2)$ from equation (55) such that

$$C_{j+1} - C_j < q-1 \text{ for some } j \ (1 \leq j \leq q-2). \tag{56}$$

Assume equation (56) is false. We then have

$$\begin{array}{rcl} C_{q-1} & := & \max\limits_{1 \leq i \leq q-1} \sum_{k=1}^{q-1} ka_{i,k} \ \text{ by (55)} \\ & := & \sum_{k=1}^{q-1} ka_{i_0,k} \ \text{ for some } \ i_0 \ (1 \leq i_0 \leq q-1) \\ & = & C_1 + \sum_{k=1}^{q-2} (C_{k+1} - C_k) \\ & \geq & C_1 + \sum_{k=1}^{q-2} (q-1) \ \text{ by assumption} \\ & = & C_1 + (q-2)(q-1) \\ & > & (q-2)(q-1) \ \text{ by (55).} \end{array} \tag{57}$$

On the other hand, we estimate C_{q-1} from equations (27) and (50). Since each nonzero $a_{i_0,k}$ $(0 \leq i_0 \leq q-1)$ is even ≥ 2 from equation (29) where $a_{i_0,k}$ $(0 \leq i_0 \leq q-1)$ are given by the second line of equation (57), there are at most $(p-1)/2$ -numbers of nonzero $a_{i_0,k} \geq 2$ $(0 \leq k \leq q-1)$. Notice that each nonzero $a_{i_0,k}$ is a small even number due to equations (36) and (39) with $2 \leq a_{i_0,k} \leq 2q_0$ $(0 \leq k \leq q-1)$. It follows that there are at least $(q - (p-1)/2)$ -numbers of $a_{i_0,k} = 0$ $(0 \leq k \leq q-1)$. We then have

$$
\begin{array}{lll}
C_{q-1} = & \sum_{k=0}^{q-1} k a_{i_0,k} & \\
= & \sum_{k=0}^{q-1} i_0 k \pmod q \, a_{1,k} & \text{by (34)} \\
= & \sum_{l=1}^{q_0} 2l(\sum_{k \in S_{1,l}} i_0 k \pmod q) & \text{by (50)} \\
< & (\sum_{l=1}^{q_0} 2l \mid S_{1,l} \mid)(q-1) & \\
= & (p-1)(q-1) & \text{by (39)} \\
< & (q-2)(q-1). &
\end{array} \tag{58}
$$

The last line of inequality (58) follows from $p-1 < q-2$ since $p+1 < q$. Inequality (58) contradicts inequality (57). This establishes inequality (56). For j chosen from inequality (56), since each nonzero $a_{i,k} \geq 2$ $(1 \leq i \leq q-1,\ 0 \leq k \leq q-1)$ from equation (29), we then have:

$$
2 \leq (C_j,\ C_{j+1}) = (C_j,\ C_{j+1} - C_j) < q-1. \tag{59}
$$

Equation (59) implies $C_j := \sum_{k=0}^{q-1} k a_{u,k}$ and $C_{j+1} := \sum_{k=0}^{q-1} k a_{v,k}$ for some u, v $(1 \leq u,\ v \leq q-1)$, have no common factor q, which leads to $q \nmid \sum_{k=0}^{q-1} k a_{1,k}$ in view of equation (34), thereby proving equation (53). Consequently, each $\sum_{k=0}^{q-1} k a_{i,k}$ $(1 \leq i \leq q-1)$ has no factor q from equations (34), (53) and Lemma 3. We thus have:

$$
\sum_{k=0}^{q-1} k a_{i,k} \ \not\equiv\ 0 \bmod q,\ 1 \leq i \leq q-1. \tag{60}
$$

Equation (60) is equivalent that the map:

$$
\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1} \longmapsto \mathbb{Z}/q\mathbb{Z}
$$

is an isomorphism. Furthermore equations (49) and (60) reveal the structure of the additive group $\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1}$ which is nontrivial, namely

$$
\sum_{k=0}^{q-1} k a_{i,k} + \sum_{k=0}^{q-1} k a_{q-i,k} \equiv\ 0 \bmod q,\ 1 \leq i \leq q-1. \tag{61}
$$

Equations (60) and (61) show $\sum_{k=0}^{q-1} k a_{i,k} \bmod q$ and $\sum_{k=0}^{q-1} k a_{q-i,k} \bmod q$ are additive inverse to each other modulo q for each $i = 1, 2, \ldots q-1$. Clearly $\sum_{k=0}^{q-1} k a_{0,k} = 0 \equiv$

$0 \pmod q$ is the additive identity modulo q from equation (30). This completes the proof of Lemma 4.

Since $p \nmid A(p)$ from equation (12), conjunction of Lemma 3 and Lemma 4 leads us to:

Corollary 5 *Let* $691 \mid A(p)$. *An odd prime* q *divides both* $A(p)$ *and* $B(p)$ *only if* $q < p$.

¿From Lemma 4, we have in particular for $i = 1$:

$$B(p) = 691 \sum_{j=1}^{p-1} \sigma_5(j)\sigma_5(p-j) \equiv \sum_{k=0}^{q-1} k a_{1,k} \not\equiv 0 \bmod q \text{ by (33) \& (60).} \tag{62}$$

Equation (62) implies $q \nmid B(p)$ and hence $A(p) \neq B(p)$ and $\tau(p) = (A(p) - B(p))/3 \neq 0$ via the unique factorization theorem if $691 \mid A(p)$. If $691 \nmid A(p)$, then since $691 \mid B(p)$ from equation (8), we trivially have $A(p) \neq B(p)$ and $\tau(p) = (A(p) - B(p))/3 \neq 0$ via the unique factorization theorem in this case too. We thus have:

Theorem 6 $\tau(p) \neq 0$ *for each prime* p.

For $691 \mid A(p)$ and $q \mid A(p)$ with $q > p$, conjunction of Lemma 3 (statement (v)) and Lemma 4 implies:

Corollary 7 *Suppose* p *satisfies equation* (10) *or* (11). *Let* $q \mid A(p)$ *with* $q > p$. *Then*

$$\min_{1 \leq i < j \leq q-1} \left(\sum_{k=0}^{q-1} k a_{i,k}, \sum_{k=0}^{q-1} k a_{j,k}\right) = 2.$$

Now let $\alpha \geq 2$. Then equations (10) and (11) are no longer equivalent. As in the case of $\alpha = 1$, since $A(p^\alpha) \equiv 3 \bmod p^5$ and $p^{11\alpha-1} < A(p^\alpha) < p^{11\alpha}$ from equation (8), an almost identical proof of Lemma 2 works for $\alpha \geq 2$, where in equation (14), the upper limit for the sum is replaced by $11\alpha - 1$. We thus have:

Lemma 8 *Let* $691 \mid A(p^\alpha)$ *for* $\alpha \geq 2$. *There is at least one prime* $q \mid A(p^\alpha)$ *with* $q > p^\alpha$.

For $q \mid A(p^\alpha)$ with $q > p^\alpha$, construct matrix $[a_{i,k}]_{0 \leq i,\, k \leq q-1}$ exactly the same way as in equation (27). Then properties (29)$-$(35), (37)$-$(41) hold with p replaced by p^α. Likewise almost identical proof of Lemma 4 works for $\alpha \geq 2$. We thus have:

Lemma 9 *Let* $691 \mid A(p^\alpha)$ *for* $\alpha \geq 2$. *Let* $q \mid A(p^\alpha)$ *with* $q > p^\alpha$. *Then* $\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1}$ *forms an additive group of order* q *modulo* q.

In particular for $i = 1$ from Lemma 9 and equation (33), we have for $\alpha \geq 2$

$$B(p^\alpha) = 691 \sum_{j=1}^{p^\alpha - 1} \sigma_5(j)\sigma_5(p^\alpha - j) \equiv \sum_{k=0}^{q-1} k a_{1,k} \not\equiv 0 \mod q. \tag{63}$$

Equation (63) implies $q \nmid B(p^\alpha)$ and hence $A(p^\alpha) \neq B(p^\alpha)$ and $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \neq 0$ by the unique factorization theorem. If $691 \nmid A(p^\alpha)$, since $691 \mid B(p^\alpha)$ from equation (8), we then trivially have $A(p^\alpha) \neq B(p^\alpha)$ and $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \neq 0$ via the unique factorization theorem in this case too. We thus have:

Theorem 10 $\tau(p^\alpha) \neq 0$ *for each* $\alpha \geq 2$.

Finally we show $\tau(n) \neq 0$ for any positive integer n.

Theorem 11 *(Lehmer's Conjecture)* $\tau(n) \neq 0$ *for each* $n \geq 1$.

Proof. Since $\tau(1) = 1$, it suffices to prove the theorem when n is composite from Theorem 6 and Theorem 10. Write

$$n = p_0^{s_0} p_1^{s_1} \dots p_u^{s_u},\ p_0 := 2,\ s_0 \geq 0,\ s_j \geq 1,\ 1 \leq j \leq u.$$

Since $\tau(n)$ is multiplicative ([1 : 92 − 93], [2 : 52 − 53], [4 : 122], [5], [6]), Theorem 11 readily follows from Theorem 6 or Theorem 10, namely

$$\begin{array}{rcl} \tau(n) & = & \prod_{j=0}^{u} \tau(p_j^{s_j}) \\ & \neq & 0. \end{array} \tag{64}$$

This completes the proof.

Suppose for each $\alpha \geq 1$,

$$A(p^{\alpha}) \equiv 0 \mod 691. \tag{65}$$

Equation (65) is equivalent to

$$p^{\alpha+1} \equiv 1 \mod 691 \text{ and } (p-1,\ 691) = 1. \tag{66}$$

Equation (66) implies the following periodicity theorem modulo 691:

Theorem 12 *(periodicity modulo* 691*) Suppose* $691 \mid A(p^{\alpha})$ *for* $\alpha \geq 1$*. Then we have*

$$A(p^{\alpha+k(\alpha+1)}) \equiv 0 \mod 691,\ k = 0, 1, 2, \ldots$$

The values of α satisfying the periodicity of $A(p^{\alpha}) \equiv 0 \bmod 691$ for each $\alpha \geq 1$ have gaps in view of equation (66) and Fermat's little theorem, namely $A(p^{\alpha}) \not\equiv 0 \bmod 691$ if and only if the factors of $\alpha + 1$ do not divide $690 = 2.3.5.23$. Thus $A(p^{\alpha}) \not\equiv 0 \bmod 691$ for α in the following set S of numbers:

$$S := \{6, 10, 12, 16, 18, 28, 30, 36, 40, 42, 46, 48, 52, 58, \ldots\}.$$

Needless to say $A(p^\alpha) \neq B(p^\alpha)$ and hence $\tau(p^\alpha) \neq 0$ for each $\alpha \in S$ by equation (8) with the aid of the unique factorization theorem.

Remark 13 *For an odd prime $q \mid A(p^\alpha),\ \alpha \geq 1$ with $q < p^\alpha$, as long as $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ forms an additive group of order q modulo q, then $q \nmid B(p^\alpha)$ by Lemma 4 or Lemma 9. It follows that $A(p^\alpha) \neq B(p^\alpha)$ and hence $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \neq 0$ in this case too. For $691 \mid A(p)$, computer simulation reveals $A(p)$ has at least one odd prime factor $q \neq 691,\ q \mid A(p)$ with $q < p$ for which $q \nmid B(p)$ for each prime $p \leq 1100000$ except $p = 186569, 290219, 464351, 671651$. Let $691 \mid A(p)$ and let $A_1(p)$ be the product of prime divisors $q \mid A(p)$ for which $q < p$ with their respective powers and $A_2(p)$ the product of prime divisors $q \mid A(p)$ for which $q > p$ with their respective powers. Computer simulation shows $C_1 p^2 < A_1(p) < C_2 p^5$ and $C_3 p^6 < A_2(p) < C_4 p^{10}$ with absolute constants $C_1, C_2, C_3, C_4 < 1$ for primes $p \leq 3000000$.*

In Table 1, we list primes p such that both 691 and q divide $A(p)$ with $q > p$ and the cardinality $\mid S_{1,l} \mid\ (1 \leq l \leq 5)$, thereby confirming inequality (41) with $q_0 \leq 4$. Notice that in Table 1, each prime p with the associated prime $q \mid A(p)$ with $q > p$, satisfies equations (38) and (39). Computer simulation reveals that a majority of respective relatively large odd prime factors less than p of both $A(p)$ and $B(p)$ are distinct. Likewise, an overwhelming majority of common odd prime factors of both $A(p)$ and $B(p)$ for which $691 \mid A(p)$ are relatively small apart from 691, thereby confirming Corollary 5. In Table 2, we list primes $p \leq 3000000$ such that $691 \mid A(p)$ and the maximum odd prime factors of $(A(p),\ B(p))$ other than 691 are ≥ 11.

Acknowledgment. We are deeply grateful to the referee who pointed out the obscurity of the additive group structure of $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ in our original manuscript. We are thankful to S. Durbha, J. Gerver, H. Li and M. Nerurkar for many lively discussions.

We are grateful to D. Zeilberger for his encouragement. The constructive criticisms of P. Deligne ([9]) and D. Bao for the proof of Lemma 2 is gratefully acknowledged.

Table 1

p	q	$\mid S_{1,0} \mid$	$\mid S_{1,1} \mid$	$\mid S_{1,2} \mid$	$\mid S_{1,3} \mid$	$\mid S_{1,4} \mid$	$\mid S_{1,5} \mid$
8291	216113	212008	4065	40	0	0	0
29021	1357091	1342657	14358	76	0	0	0
30403	1283839	1268731	15015	93	0	0	0
34549	789673	772578	16918	175	2	0	0
51133	112919	89995	20474	2267	174	9	0
53897	371549	345582	25014	925	28	0	0
96739	392957	347376	42917	2543	118	3	0

Table 2

p	$(A(p),\ B(p))$
547271	2.3.11.691
610843	2.3.17.691
988129	2.3.5.13.691
1112509	2.3.5.23.691
1336393	2.3.101.691
1405493	2.3.113.691
1716463	2.3^2.23.691
1875373	2.23.691
1940327	$2^2.3^2$.13.691
2126897	2.3^3.19.691
2128279	2^2.5.11.691
2161447	2^2.23.691
2198761	2.43.691
2447521	2.23.691
2479307	2.23.691
2538733	2.11.691
2542879	2^4.3.5.23.691
2956097	2.23.691

References

1. Apostol, Tom, Modular Functions And Dirichlet Series, Springer (1997).

2. Berndt, B., Number Theory in the Spirit of Ramanujan, AMS (2006).

3. Iwaniec, H., Topics in Classical Automorphic Forms, AMS (1997), $13-22$.

4. Koblitz, N., Int. to Elliptic Curves And Modular Functions, Springer (1993), $108-123$.

5. Lehmer, D.H., Ramanujan's Function $\tau(n)$, Duke Math. J. 10 (1943), $483-492$.

6. Lehmer, D.H., The Vanishing of Ramanujan's Function $\tau(n)$, Duke Math. J. 14 (1947), $429-433$.

7. Lehmer, D.H., Note on the Distribution of Ramanujan's τ Function, Math. Comp.24 (1970), $741-743$.

8. Hua, L.K., Int. to Number Theory, Springer (1982) $204-216$.

9. Deligne, P., Personal Correspondence.

Rutgers University-Camden

Camden, NJ 08102 USA